The Restricted Value Function of MILPs and its construction with SYMPHONY

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Overview

Typically, a mathematical optimization model assumes that a **decision** must be taken:

- by a single decision maker;
- in a single point in time;
- under perfect information;
- with the goal of optimizing a **single objective**.

However, in most real-world applications these assumptions do not hold!

In these scenarios, there is a **functional dependence** on various components of the input:

- **Multi-level Optimization** $\rightarrow$ followers' reaction impact to the leader's decision;
- **Multi-stage Optimization** $\rightarrow$ each stage's decisions dependence on future stages under uncertainty;
- **Multi-objective Optimization** $\rightarrow$ trade-offs in optimizing multiple criteria simultaneously.

These dependencies can be all captured by the so-called **Value Function**.

The Value Function of a Mixed-Integer Linear Problem

The **Value Function** (VF) of a Mixed-Integer Linear Problem (MILP) describes how the **optimal objective value** varies when the **right-hand side** (RHS) changes.

Sensitivity analysis and Warm-starting for MILPs [Ralphs and Güzelsoy, 2004]

Improved algorithmic performance when solving a sequence of related MILPs.

Optimality conditions and valid inequalities [Bolusani and Ralphs, 2022]

VF provides optimality conditions for the follower's problem in **Bilevel Optimization**, which can be exploited to generate valid inequalities.

Decomposition methods [Hassanzadeh and Ralphs, 2014]

Benders' algorithm for **Two-stage stochastic Optimization** and **Bilevel Optimization**.

Construction of the Efficient Frontier [Fallah et al., 2024]

The **restricted VF** contains the Efficient Frontier of a **Multi-Objective MILP** and can be constructed in a single branch-and-bound tree.

A Use Case: Multi-Objective Mixed-Integer Linear Optimization

A general **Multi-Objective Mixed-Integer Linear Problem** (MO-MILP) is defined by

$$\operatorname{vmin}\left\{(c^0 x, Cx) \;\middle|\; x \in \mathcal{F}\right\}, \qquad \text{(MO-MILP)}$$

where

- $\mathcal{F} = \{x \in X \mid Ax = b\}$ and $A \in \mathbb{Q}^{k \times n}$; $b \in \mathbb{Q}^k$; $X \subseteq \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+$;
- $c^0 \in \mathbb{Q}^n$; $C \in \mathbb{Q}^{\ell \times n}$; $L = \{1, \ldots, \ell\}$.
- **Assumption**: $\mathcal{F}$ is non-empty and bounded.

Decision space: n-dimensional space containing the feasible solutions $\mathcal{F}$.
Criterion space: $(\ell + 1)$-dimensional space containing the images of $\mathcal{F}$.

Nondominated Points

Given $\bar{x} \in \mathcal{F}$, the image $(c^0\bar{x}, C\bar{x})$ is a **nondominated point** (NDP) if

$$\nexists x \in \mathcal{F}, \text{ with } x \neq \bar{x}, c^0 x \geq c^0 \bar{x} \text{ and } (Cx)_j \geq (C\bar{x})_j, \text{ for all } j \in L; \text{ and}$$

$$\text{either } c^0 x > c^0 \bar{x} \text{ or } (Cx)_j > (C\bar{x})_j, \text{ for some } j \in L.$$

Solving (MO-MILP) means describing the **Efficient Frontier** (EF), i.e., the set of NDPs.

The Restricted Value Function

Let us consider a single-objective reformulation of (MO-MILP) in which:

- all-but-one criteria $\left\{C^1x, \dots, C^\ell x\right\}$ are imposed as constraints;
- we make such constraints **parametric**, while keeping the other fixed.

Then, the **Restricted Value Function** is the function $\phi_L : \mathbb{R}^\ell \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\phi_L(\zeta) = \min\left\{c^0x \;\middle|\; x \in \mathcal{S}(\zeta)\right\}, \tag{RVF}$$

where $\mathcal{S}(\zeta) = \{x \in X \mid Cx \leq \zeta, Ax = b\}$, for $\zeta \in \mathbb{R}^\ell$. We set $\phi_L(\zeta) = \infty$, if $\mathcal{S}(\zeta) = \emptyset$.

The epigraph of ϕ_L is defined by

$$epi(\phi_L) = \left\{(\tau, \zeta) \in \mathbb{R} \times \mathbb{R}^\ell \;\middle|\; \phi_L(\zeta) \leq \tau\right\}. \tag{1}$$

- Note that, for all $x \in \mathcal{F}$, the image $(c^0x, Cx) \in \text{epi}(\phi_L)$.
- In particular, if (c^0x, Cx) is an NDP, then $c^0x = \phi_L(Cx)$.

Theorem [Fallah et al., 2024]

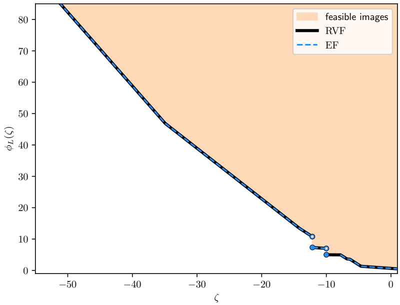
The EF of (MO-MILP) is a (possibly strict) subset of the boundary of $\text{epi}(\phi_L)$.

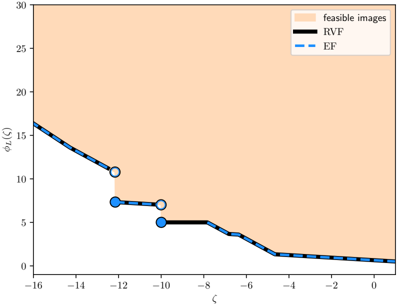
An Example: The RVF and EF

$$\begin{aligned}
\phi_L(\zeta) = \min \quad & 2x_1 + 5x_2 + 7x_4 + 10x_5 + 2x_6 + 10x_7 \\
\text{s.t.} \quad & -x_1 - 10x_2 + 10x_3 - 8x_4 + x_5 - 7x_6 + 6x_7 \leq \zeta \\
& -x_1 + 4x_2 + 9x_3 + 3x_4 + 2x_5 + 6x_6 - 10x_7 = 4 \\
& x_4 + 5x_2 \leq 5 \\
& x_7 + 5x_2 \leq 5 \\
& x_j \in \{0,1\}, \quad \forall j \in \{1,2\} \\
& x_j \in \mathbb{R}_+, \quad \forall j \in \{3,4,\ldots,7\}.
\end{aligned}$$

Bi-objective (objective and first constraint)

$\mathcal{F}$ (remaining constraints)





On the Construction of the Restricted Value Function

- We want to find a description of the RVF, since it provides proofs for points to be on the **boundary** of epi(ϕ_L).
- Such proofs have several important interpretations:
  - **Multi-objective Optimization** $\rightarrow$ a point belongs to the EF;
  - **Multi-level Optimization** $\rightarrow$ a follower's response is optimal for the leader's decision;
  - **Multi-stage Optimization** $\rightarrow$ an early-stage decision is optimal for future stages.
- Generally, it is easier to construct **lower approximations** of the RVF.
- **General Duality** is the theory that defines and construct such approximations in terms of **dual functions** of ϕ_L.
- Dual functions arise in many solution frameworks, specifically **branch-and-bounds**.
- We can iteratively refine a **lower approximation** of the RVF, by evaluating it using a single branch-and-bound tree [Hassanzadeh and Ralphs, 2014].

Duality Theory for the Restricted Value Function

The **General Dual Problem** [Tind and Wolsey, 1981] associated to (RVF) for a specific $\hat{\zeta} \in \mathbb{R}^\ell$ is

$$\max \left\{ F(\hat{\zeta}) \;\middle|\; F(d) \leq \phi_L(d), \forall d \in \mathbb{R}^\ell, F \in \Upsilon^\ell \right\}, \tag{2}$$

where $\Upsilon^\ell \subseteq \left\{ f \;\middle|\; f : \mathbb{R}^\ell \to \mathbb{R} \right\}$. Some remarks:

- Feasible solutions to (2) are **dual functions**, i.e., lower approximations of ϕ_L.
- **Strong duality** holds if $\phi_L \in \Upsilon^\ell$.

- Let $\hat{\tau} \in \mathbb{R}$ be given. Does $(\hat{\tau}, \hat{\zeta})$ lie on the boundary of $\text{epi}(\phi_L)$?
- Suppose F^* is an **optimal dual function** to (2). If $F^*(\hat{\zeta}) = \hat{\tau}$, then F^* certifies that

$$\exists x \in \mathcal{F} \text{ such that } Cx = \hat{\zeta} \text{ and } c^0 x = \hat{\tau} = F^*(\hat{\zeta}) = \phi_L(\hat{\zeta})$$

- Therefore, F^* certifies that $(\hat{\tau}, \hat{\zeta})$ lies on the boundary of $\text{epi}(\phi_L)$.
- In this case, F^* is a **strong dual function** at $\hat{\zeta}$.

Theorem

If $\exists$ a dual function $F^* \in \Upsilon^\ell$ optimal for (2) such that $F^*(\hat{\zeta}) = \hat{\tau}$, then $(\hat{\tau}, \hat{\zeta})$ lies on the boundary of $\text{epi}(\phi_L)$.

Dual Approximations of the Restricted Value Function

Any B&B tree used to evaluate $\phi_L(\zeta)$ encodes a **dual function** of the RVF/EF.

- Let T be the index set of the terminating nodes of the tree.
- The RVF of the **LP relaxation** at node $t \in T$ is

$$\begin{aligned} \phi_L^t(\zeta) = \min\ & c^0 x \\ \text{s.t. } & Cx \leq \zeta \\ & Ax = b \\ & l^t \leq x \leq u^t, x \geq 0 \end{aligned} \tag{BB.LP.P}$$

- By LP duality, we have that:

$$\begin{aligned} \phi_L^t(\zeta) = \max\ & v\zeta + wb + \underline{\pi} l^t + \overline{\pi} u^t \\ \text{s.t. } & vC + wA + \underline{\pi} + \overline{\pi} \leq c^0 \\ & v, \overline{\pi} \leq 0, \underline{\pi} \geq 0 \end{aligned} \tag{BB.LP.D}$$

Given any collection D of solutions feasible to (BB.LP.D), we obtain that the function

$$F(\zeta) = \min_{t \in T} \max_{(v,w,\underline{\pi},\overline{\pi}) \in D} v\zeta + wb + \underline{\pi} l^t + \overline{\pi} u^t, \qquad \forall \zeta \in \mathbb{R}^\ell, \tag{3}$$

is dual to both the RVF and the EF.

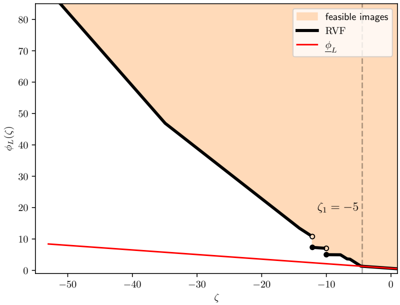
A finite Algorithm for constructing the Restricted Value Function

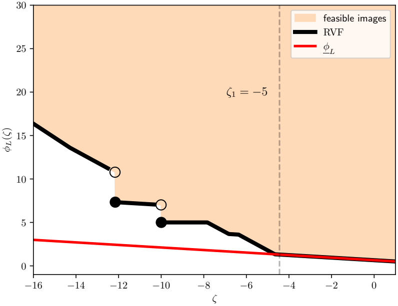
- Evaluate the RVF at a sequence of RHSs ζ's using the branch-and-bound algorithm.
- When the RHS change, keep branching in the same tree, updating T.
- Maintain and update the collection D of dual solutions generated by solving the LP relaxation.
- There is a finite sequence of ζ's for which the algorithm converges finitely to the exact RVF.
- The key for the efficiency of this algorithm is finding the right set of ζ's.

An Example: Evolution of the Dual Function

Node 0

$\underline{\phi}_L^0 = v^1 \zeta + \alpha^1$

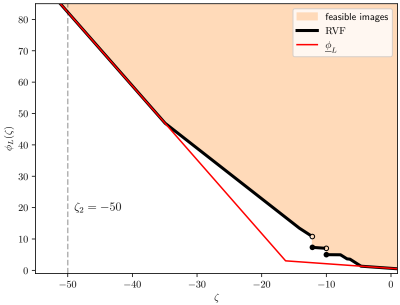


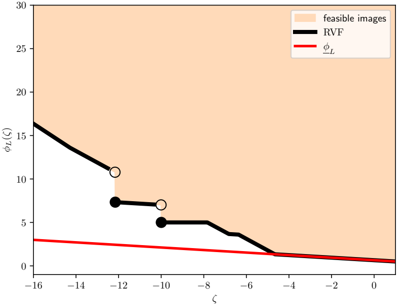


An Example: Evolution of the Dual Function

Node 0

$$\underline{\phi}_L^0 = \max\left\{v^1\zeta + \alpha^1, v^2\zeta + \alpha^2\right\}$$

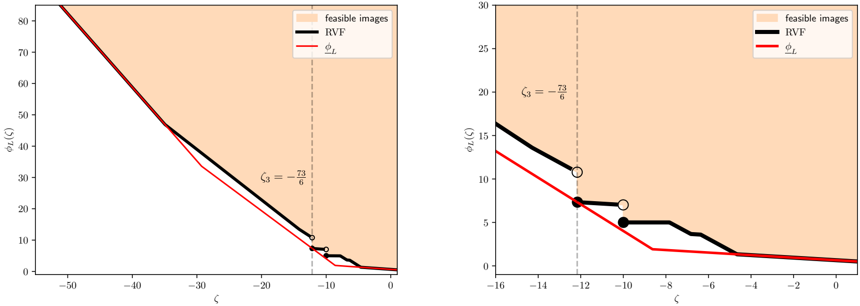




An Example: Evolution of the Dual Function

Node 0

$$\underline{\phi}_L^0 = \max\left\{v^1\zeta + \alpha^1, v^2\zeta + \alpha^2, v^3\zeta + \alpha^3\right\}$$



Node 0

$$\underline{\phi}_L^0 = \max\left\{v^1\zeta + \alpha^1,\, v^2\zeta + \alpha^2,\, v^3\zeta + \alpha^3,\, v^4\zeta + \alpha^4\right\}$$

$x_1 \leq 0$ → Node 1; $x_1 \geq 1$ → Node 2

Node 1

$$\underline{\phi}_L^1 = \max\left\{\underline{\phi}_L^0,\, v^5\zeta + \alpha^5\right\}$$

$x_2 \leq 0$ → Node 4; $x_2 \geq 1$ → Node 5

Node 2

$$\underline{\phi}_L^2 = \max\left\{\underline{\phi}_L^0,\, v^6\zeta + \alpha^6\right\}$$

$x_2 \leq 0$ → Node 6; $x_2 \geq 1$ → Node 7

Node 4

$$\underline{\phi}_L^4 = \max\left\{\underline{\phi}_L^1,\, v^7\zeta + \alpha^7\right\}$$

Node 5

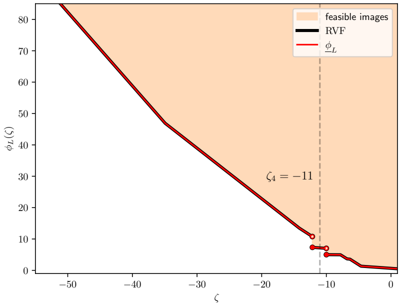
$$\underline{\phi}_L^5 = \max\left\{\underline{\phi}_L^1,\, v^8\zeta + \alpha^8\right\}$$

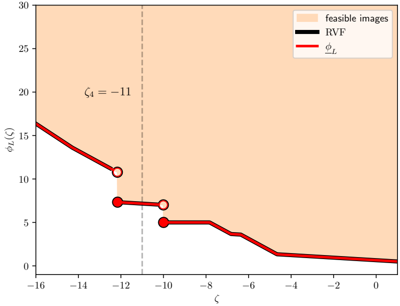
Node 6

$$\underline{\phi}_L^6 = \max\left\{\underline{\phi}_L^2,\, v^9\zeta + \alpha^9\right\}$$

Node 7

$$\underline{\phi}_L^7 = \max\left\{\underline{\phi}_L^2,\, v^{10}\zeta + \alpha^{10}\right\}$$





SYMPHONY

- SYMPHONY is an open source MILP solver framework with unique capabilities
  - Can warm-start solution of a modified instance in the same tree.
  - Can explicitly build dual functions from B&B trees.
  - Can be used to construct the Value Function and the Efficient Frontier.
- Its infrastructure makes the Algorithm for the EF easy to implement.
- SYMPHONY is also used in the **Bilevel MILP** solver MibS, and can be used to warm-start sequence of related MILPs arising in this setup (e.g., feasibility check).
- A generalized Benders' algorithm for **Two-stage Stochastic** MILPs with recourse is also being revived.

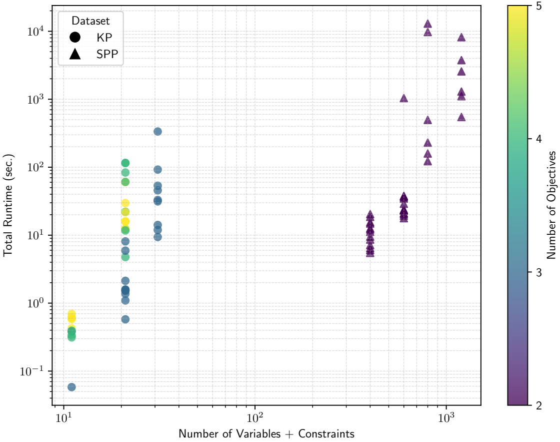
Numerical Results

Instances:

- MO Knapsack Problem (KP);
- MO Set Partitioning Problem (SPP).

Setup:

- CPU: Apple M2 Pro;
- Time Limit: 4 hours.



Final Remarks

Core concepts

- The **Restricted Value Function** encodes optimality conditions for MILPs.
- **Duality Theory** provides certificates of optimality by mean of **strong dual functions**
- There exists a single B&B tree whose **dual function** coincide to the RVF.

These concepts can serve as the foundation of a new class of algorithms for

- **Multi-objective Optimization**;
- **Multi-level Optimization**;
- **Multi-stage Optimization**.

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- **Multi-stage Optimization**.

Thanks for your attention!

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