Exploiting Dual Functions in Mixed Integer Bilevel Linear Programs

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Mixed Integer Linear Bilevel Optimization

- **Bilevel Problems** (BP) involve optimizing the strategy of a leader and the subsequent reaction of a follower:

$$\begin{aligned} \min\ & cx + d^1 y \\ \text{s.t.}\ & A^1 x + G^1 y \geq b^1 \\ & x \in X \\ & y \in \operatorname{argmin} \left\{ d^2 y \ : G^2 y \geq b^2 - A^2 x,\ y \in Y \right\} \end{aligned} \tag{BP}$$

- Stackelberg game: two-players sequential game
- Leader's decision must take into account follower's optimal reaction
- $x \in X \subseteq \mathbb{Z}_+^{r_1} \times \mathbb{R}_+^{n_1 - r_1}$ are controlled by the leader
- $y \in Y \subseteq \mathbb{Z}_+^{r_2} \times \mathbb{R}_+^{n_2 - r_2}$ are controlled by the follower

Input Data

1. Upper Level Problem (ULP): $c \in \mathbb{Q}^{n_1}, d^1 \in \mathbb{Q}^{n_2}, A^1, G^1 \in \mathbb{Q}^{m_1 \times n_1}, b^1 \in \mathbb{Q}^{m_1}$
2. Second Level Problem (SLP): $d^2 \in \mathbb{Q}^{n_2}, A^2, G^2 \in \mathbb{Q}^{m_2 \times n_1}, b^2 \in \mathbb{Q}^{m_2}$

Value Function Reformulation

Definition

Let $\phi : \mathbb{R}^{m_2} \to \mathbb{R} \cup \{\pm\infty\}$ be a function such that

$$\phi(\beta) = \min\left\{d^2 y \;:\; G^2 y \geq \beta,\; y \in Y\right\}.$$

We assume

$$\phi(\beta) = +\infty, \text{if (SLP) is } \textbf{infeasible},$$

$$\phi(\beta) = -\infty, \text{if (SLP) is } \textbf{unbounded},$$

for some $\beta \in \mathbb{R}^{m_2}$.

- Then (BP) becomes:

$$\begin{aligned} \min\;& cx + d^1 y \\ \text{s.t. }& A^1 x + G^1 y \geq b^1 \\ & A^2 x + G^2 y \geq b^2 \\ & d^2 y \leq \phi(b^2 - A^2 x) \\ & x \in X,\; y \in Y \end{aligned} \tag{BP-VF}$$

On Solving MILBP

Relaxation

In a Branch-and-Bound-or-Cut framework, an LP relaxation of (BP-VF) is considered at each node by dropping:

1. x and y integrality
2. Second level optimality $d^2 y \leq \phi(b^2 - A^2 x)$

Bilevel feasibility

Checking feasibility of an integer point $(\hat{x}, \hat{y})$ involves the evaluation of

$$d^2 \hat{y} \leq \phi(b^2 - A^2 \hat{x})$$

which is an **NP-Hard** task.

Complexity

(BP) is Σ_2^p-hard $\Rightarrow$ Given an oracle for $\phi(\beta)$, (BP) requires **nondeterministic polynomial time** to be solved.

About this Work

We consider two open-source solvers:

MibS [Tahernejad, Ralphs, and DeNegre 2020]

- Branch-and-Cut for (BP)
- Checking feasibility using MILP solvers

SYMPHONY [T. K. Ralphs and Güzelsoy 2005]

- Implementing Branch-and-Cut solver for MILPs (SLP)
- Warm-starting capabilities for RHS changes

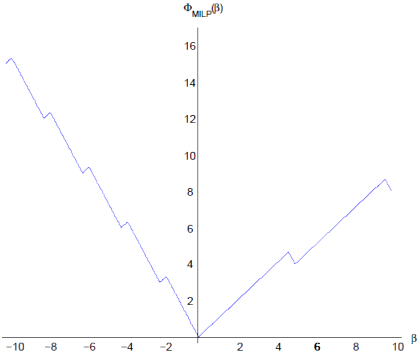
Our contribution

1. How to construct an iteratively refined approximation of ϕ using SYMPHONY
2. How to use it to efficiently evaluate $d^2 y \leq \phi(b^2 - A^2 x)$ to improve the feasibility check

On the MILP Value Function

Let us consider the following (SLP):

$$\begin{aligned} \phi(\beta) = \min \quad & 6y_1 + 4y_2 + 3y_3 + 4y_4 + 5y_5 + 7y_6 \\ \text{s.t.} \quad & 2y_1 + 5y_2 - 2y_3 - 2y_4 + 5y_5 + 5y_6 = \beta \\ & y_1, y_2, y_3 \in \mathbb{Z}_+, y_4, y_5, y_6 \in \mathbb{R}_+ \end{aligned}$$



Some properties

ϕ of a MILP is:

- Lower semi-continuous
- Subadditive
- Piecewise polyhedral

On the MILP Value Function

A description of ϕ

Let I and C index the integer and continuous (resp.) variables in (SLP). As noted in [T. K. Ralphs and Hassanzadeh 2014], we have that

$$\phi(\beta) = \min_{y_I \in \mathbb{Z}_+^{r_2}} \{\phi_I(G_{2I} y_I) + \phi_C(\beta - G_{2I} y_I)\}.$$

Integer Restriction:

$$\phi_I(\gamma) = \min_{y_I \in \mathbb{Z}_+^{r_2}} \left\{ {d_2}_I^\top y_I \mid G_{2I} y_I = \gamma \right\}$$

Continuous Restriction:

$$\phi_C(\gamma) = \min_{y_C \in \mathbb{R}_+^{n_2 - r_2}} \left\{ {d_2}_C^\top y_C \mid G_{2C} y_C = \gamma \right\}$$

Theorem [T. K. Ralphs and Hassanzadeh 2014]

Under the assumption that $\{\beta \in \mathbb{R}^{m_2} \mid \phi_I(\beta) < \infty\}$ is finite, there exists a finite set $\mathcal{S} \subset \mathbb{Z}_+^{r_2}$ such that

$$\phi(\beta) = \min_{y_I \in \mathcal{S}} \{\phi_I(G_{2I} y_I) + \phi_C(\beta - G_{2I} y_I)\}.$$

Lower Approximations of ϕ

Dual functions [Hassanzadeh and T. K. Ralphs 2014]

A function $f : \mathbb{R}^{m_2} \to \mathbb{R} \cup \{\pm\infty\}$ is said to be dual to ϕ if

$$f(\beta) \leq \phi(\beta), \qquad \beta \in \mathbb{R}^{m_2}.$$

Moreover, f is strong at $\hat{\beta} \in \mathbb{R}^{m_2}$ is $f(\hat{\beta}) = \phi(\hat{\beta})$.

Strong dual functions usages examples:

- warm-start MILP resolution for different RHS $\beta' \in \mathbb{R}^{m_2}$
- sensitivity analysis
- **optimality proofs**

Strong Dual Functions from a Branch-and-Bound Tree

- Let $\hat{\beta} \in \mathbb{R}^{m_2}$ be such that $\phi(\hat{\beta}) < \infty$
- Let T indexing the set of leaf nodes of the optimal B&B tree for (SLP)
- For $t \in T$, let
  1. $l^t, u^t \in \mathbb{Z}^{r_2}$ be the branching bounds
  2. $(\pi^t, \underline{\pi}^t, \overline{\pi}^t) \in \mathbb{R}^{m_2+2n_2}$ be the optimal dual solution
- Then the LP primal-dual pair is

$$\min_{y \in \mathbb{R}^{n_2}_+} \{d^2 y \mid G^2 y = \hat{\beta},\ l^t \leq y \leq u^t\} \qquad (P^t)$$

$$\max_{(\pi,\underline{\pi},\overline{\pi}) \in \mathcal{D}} \{\hat{\beta}\pi + l^t \underline{\pi} - u^t \overline{\pi}\} \qquad (D^t)$$

$$\mathcal{D} = \left\{(\pi, \underline{\pi}, \overline{\pi}) \in \mathbb{R}^{m_2+2n_2} \mid G^2\pi + \underline{\pi} - \overline{\pi} \leq d^2, \underline{\pi} \geq 0, \overline{\pi} \geq 0\right\}$$

- $\mathcal{D}$ is independent from $\hat{\beta}, l^t, u^t$, then by LP duality

$$\max_{(\pi,\underline{\pi},\overline{\pi}) \in \mathcal{D}} \{\beta\pi + l^t \underline{\pi} - u^t \overline{\pi}\} \leq \min_{y \in \mathbb{R}^{n_2}_+} \{d^2 y \mid G^2 y = \beta,\ l^t \leq y \leq u^t\}, \forall\ \beta \in \mathbb{R}^{m_2}$$

Strong Dual Functions from a Branch-and-Bound Tree

Theorem [Wolsey 1981]

The function $\underline{\phi}$ defined by

$$\underline{\phi}(\beta) = \min_{t \in T} \left(\beta \pi^t + l^t \underline{\pi}^t - u^t \overline{\pi}^t \right) \qquad \forall \beta \in \mathbb{R}^{m_2},$$

is dual to ϕ and strong at RHS $\hat{\beta}$.

Warm-starting with SYMPHONY at a glance

Given a B&B tree evaluating $\phi(\hat{\beta})$, we can warm-start the solution of $\phi(\tilde{\beta})$ by:

1. Re-optimize the LP duals each at leaf nodes $t \in T$
2. Keep branching on nodes with fractional solutions (if any)

Remark

The warm-started B&B encodes a dual solution strong at $\tilde{\beta}$, but it **might not** be strong at $\hat{\beta}$ anymore. (Dual solutions at leaf nodes may change).

Maintaining a Strong Dual Function

- A stronger dual function can be obtained considering dual solutions from **all nodes** in the tree

- Given any B&B tree, let:
  1. $D \subseteq \mathcal{D}$ be its set of dual solutions generated over all nodes in the B&B (including re-optimizations)
  2. T be the set of the leaf nodes

A stronger dual function

$$\underline{\phi}^{+}(\beta) = \min_{t \in T} \max_{(\pi,\underline{\pi},\overline{\pi}) \in D} \left(\beta^{\top}\pi + l^{t}\underline{\pi} - u^{t}\overline{\pi}\right)$$

- This new dual function is strong at all RHSs previously solved and

$$\underline{\phi}(\beta) \leq \underline{\phi}^{+}(\beta) \leq \phi(\beta)$$

Evaluating $\underline{\phi}^+(\beta)$

Given any $\beta \in \mathbb{R}^{m_2}$, let us consider the following matricial representation of $\underline{\phi}^+$:

$$M_D = \begin{bmatrix} \pi^{d_1} & \underline{\pi}^{d_1} & \overline{\pi}^{d_1} \\ \pi^{d_2} & \underline{\pi}^{d_2} & \overline{\pi}^{d_2} \\ & \vdots & \\ \pi^{d_{|D|}} & \underline{\pi}^{d_{|D|}} & \overline{\pi}^{d_{|D|}} \end{bmatrix} \quad N_T = \begin{bmatrix} \beta & \beta & & \beta \\ l^{t_1} & l^{t_2} & \dots & l^{t_{|T|}} \\ u^{t_1} & u^{t_2} & & l^{t_{|T|}} \end{bmatrix}$$

Then $\underline{\phi}^+(\beta)$ can be evaluated with the following operation:

$$M_D \cdot N_T = \begin{bmatrix} \beta\pi^{d_1} + l^{t_1}\underline{\pi}^{d_1} - u^{t_1}\overline{\pi}^{d_1} & \dots & \beta\pi^{d_1} + l^{t_{|T|}}\underline{\pi}^{d_1} - u^{t_{|T|}}\overline{\pi}^{d_1} \\ \beta\pi^{d_2} + l^{t_1}\underline{\pi}^{d_2} - u^{t_1}\overline{\pi}^{d_2} & \dots & \beta\pi^{d_2} + l^{t_{|T|}}\underline{\pi}^{d_2} - u^{t_{|T|}}\overline{\pi}^{d_2} \\ \vdots & & \vdots \\ \beta\pi^{d_{|D|}} + l^{t_1}\underline{\pi}^{d_{|D|}} - u^{t_1}\overline{\pi}^{d_{|D|}} & \dots & \beta\pi^{d_{|D|}} + l^{t_{|T|}}\underline{\pi}^{d_{|D|}} - u^{t_{|T|}}\overline{\pi}^{d_{|D|}} \end{bmatrix}$$

$$\underbrace{\begin{matrix} \downarrow & & \downarrow \\ \max(\cdot) & \dots & \max(\cdot) \end{matrix}}_{\min(\cdot) = \underline{\phi}^+(\beta)}$$

Checking Bilevel Feasibility Using $\underline{\phi}^+$

Algorithm 1

Given $(\hat{x}, \hat{y}) \in X \times Y$, a B&B Tree and matrices M_D, N_T then
Evaluate $\underline{\phi}^+(b^2 - A^2\hat{x})$ using M_D, N_T
if $d^2\hat{y} \leq \underline{\phi}^+(b^2 - A^2\hat{x})$ **then**
    return $(\hat{x}, \hat{y})$ is Bilevel Feasible
else
    Warm-start the evaluation of $\phi(b^2 - A^2\hat{x})$ using SYMPHONY
    B&B Tree is updated with new leaf nodes T' and (possibly new) duals D'
    Update M_D, N_T with $M_{D'}, N_{T'}$
    if $d^2\hat{y} \leq \phi(b^2 - A^2\hat{x})$ **then**
        return $(\hat{x}, \hat{y})$ is Bilevel Feasible
    else
        return $(\hat{x}, \hat{y})$ is not Bilevel Feasible
    end if
end if

Conclusions

Remarks

- The dimension of M_D, N_T should be kept as small as possible
  - Duplicate Dual solutions should be avoided $\rightarrow$ Hash Table
  - Some Dual solutions may become dominated and may be safely discarded
- Algorithm 1 is iteratively building parts of ϕ needed to solve (BP)

Takeaways

- Dual functions can provide optimality proofs for (SLP)
- `SYMPHONY`'s warm-starting provides stronger and stronger dual functions
- Effectiveness of this approach is an empirical question

Roadmap

1. Refine the implementation
2. Parameter tuning via numerical experience