Application of $M_+(\cdot)$ operator to Representative formulation for GCP

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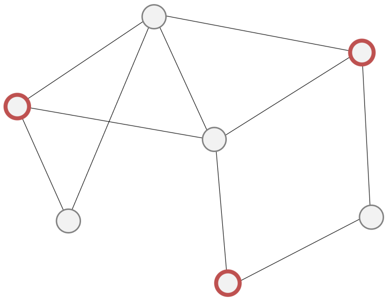
ISCO, May 2022

Some definitions

Given an undirected graph $G = (V, E)$, a set $S \subseteq V$ of pairwise non-adjacent vertices is called *stable*.



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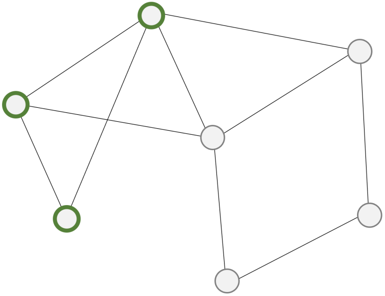
A set $K \subseteq V$ of pairwise adjacent vertices is called a *clique*.



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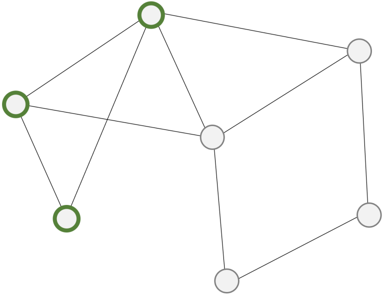
Given an undirected graph $G = (V, E)$, a set $S \subseteq V$ of pairwise non-adjacent vertices is called *stable*.
A set $K \subseteq V$ of pairwise adjacent vertices is called a *clique*.

The maximum cardinality of such a sets in G are denoted by $\alpha(G)$ and $\omega(G)$, respectively.

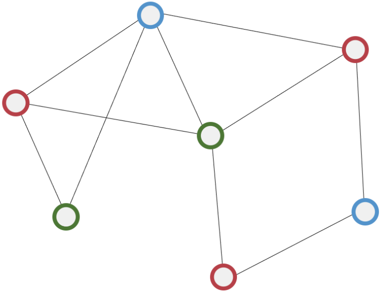


Some definitions

A *k-coloring* in G is a partition of V into k stable sets.

The minimum number k s.t. G has a k-coloring is denoted by $\chi(G)$.

The *Graph Coloring Problem* (GCP) calls for finding $\chi(G)$



Some definitions

A relaxation of the GCP is given by the so-called *fractional chromatic number* $\chi^f(G)$

$$\chi^f(G) = \min \sum_{s \in \mathcal{S}} y_s$$
$$\text{s.t.}$$
$$\sum_{s \in S(i)} y_s \geq 1, \;\; i \in V$$
$$y_s \geq 0$$

where $\mathcal{S}$ is the collection of all stable sets in G and $S(i) \subseteq \mathcal{S}$ is the subset of stable sets including vertex i

Motivation

- The computation of $\chi(G)$ and $\chi^f(G)$ are well-known to be NP-Hard[1]
- $\chi(G)$ is also hard to approximate[2]

[1] Garey and David S. Johnson 1990
[2] Khot 2001

Motivation

- The computation of $\chi(G)$ and $\chi^f(G)$ are well-known to be NP-Hard[1]
- $\chi(G)$ is also hard to approximate[2]
- Identify tight *lower bounds* of $\chi(G)$ is of importance
- Lower bounds from linear relaxations are cheap to compute but can be rather weak
- Lower bounds from *semidefinite programming* (SDP) are stronger in general but harder to handle in practice

[1] Garey and David S. Johnson 1990
[2] Khot 2001

Semidefinite Lower Bounds for GCP

SDP approaches are based on the following result:

Lovász Theta Function[1]

$$\omega(G) \leq \theta(\bar{G}) \leq \chi(G)$$

- $\bar{G}$ is the complement graph of G
- $\theta(\bar{G})$ can be computed in polynomial time via SDP
- It provides a good trade off between quality of the bound and efficiency

[1] Lovász 1979

On the Lovász Theta function

- Lovász also proved that $\theta(\bar{G}) \leq \chi^f(G)$[1]

[1] Grötschel, Lovász, and Schrijver 2012

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- Thus, the gap $\chi(G) - \theta(\bar{G})$ tends to increase as $\chi^f(G)$ gets closer to $\omega(G)$

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- Thus, the gap $\chi(G) - \theta(\bar{G})$ tends to increase as $\chi^f(G)$ gets closer to $\omega(G)$
- Improvements of $\theta(\bar{G})$ have been investigated through the addition of valid inequalities in *Szegedy 1994, Dukanovic and Rendl 2007* and more recently in *Gaar and Rendl 2020*
- $\chi^f(G)$ represents a target value not straightforward to reach with SDP

[1] Grötschel, Lovász, and Schrijver 2012

Lovász-Schrijver Lifting operator[1]

Consider the convex hull of integer solutions of some 0-1LP

$$P := conv\{x \in \{0,1\}^n \ : \ Ax \leq b\},$$

along with its continuous relaxation

$$L := \{x \in [0,1]^n \ : \ Ax \leq b\} \supseteq P.$$

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For $i = 1, \ldots, n$ generate the set of non-linear inequalities

$$\begin{aligned} x_i(Ax - b) &\leq 0 \\ (1 - x_i)(Ax - b) &\leq 0 \end{aligned} \tag{1}$$

[1] Lovász and Schrijver 1991

Lovász-Schrijver Lifting operator[1]

Linearize (1) as follows:

- replace the products $x_i x_j$ with x_{ij} and $x_i x_i$ with x_i
- Let $X \in \mathcal{S}_n$ be a symmetric, real matrix with

$$(X)_{ij} = x_{ij} \text{ and } x = \operatorname{diag}(X)$$

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$$M_+(L) := \left\{ X \in \mathcal{S}_n \;:\; (1) \text{ hold}, \; x = \operatorname{diag}(X), \; \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \right\}$$

The projection of $M_+(L)$ onto the x-space is valid for P and in general tighter than L.

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- variables: $\binom{n}{2}$
- constraints: $O(nm)$

[1] Lovász and Schrijver 1991

Lovász-Schrijver Lifting operator[1]

- Applications of this operator to 0-1LP have been investigated[2,3]
- Optimizing over $M_+(\mathsf{L})$ yields strong bounds for P in general
- A significant drawback is given by the sizes of the resulting SDPs

In this work: we investigate a new SDP relaxation obtained from the application of the Lovász-Schrijver $M_+(\cdot)$ lifting operator to a compact linear formulation for the GCP

[1] Lovász and Schrijver 1991
[2] Dash 2001
[3] Burer and Vandenbussche 2006

Representative formulation for GCP

Given a graph $G = (V, E)$, the natural LP formulation[1] assign a color to each vertex, involving $O(|V|^2)$ variables and $O(|V||E|)$ constraints.

Campêlo et al.[1] proposed a more compact formulation, in which each color class is represented by exactly one vertex, that is

$$\forall\ u \in V,\ v \in \bar{N}(u) \cup u, \text{ let } x_{uv} = \begin{cases} 1 & \text{if } u \text{ represent the color of } v \\ 0 & \text{otherwise} \end{cases}$$

where $\bar{N}(v)$ be the set of non-adjacent nodes to v in G.

[1] Méndez Dıaz and Zabala 2000

[2] Campêlo, Corrêa, and Frota 2004

Representative formulation for GCP

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$$\chi(G) = \min \sum_{u \in V} x_{uu}$$

$$\text{s.t.} \sum_{u \in \bar{N}(v) \cup v} x_{uv} \geq 1 \qquad \forall\ v \in V \tag{2}$$

$$x_{uv} + x_{uw} \leq x_{uu} \qquad \forall\ u \in V,\ (v, w) \in G[\bar{N}(u)] \tag{3}$$

$$x_{uv} \in \{0, 1\} \qquad \forall\ u \in V,\ v \in \bar{N}(u) \cup u.$$

Representative formulation for GCP

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\text{s.t.} & \sum_{u \in \bar{N}(v) \cup v} x_{uv} \geq 1 \qquad \forall\ v \in V && (2) \\
& x_{uv} + x_{uw} \leq x_{uu} \qquad \forall\ u \in V,\ (v, w) \in G[\bar{N}(u)] && (3) \\
& x_{uv} \in \{0, 1\} \qquad \forall\ u \in V,\ v \in \bar{N}(u) \cup u.
\end{aligned}$$

Idea: apply $M_+(\cdot)$ to the following polytope

$$\text{REP}(G) := \left\{ x \in [0,1]^{2|\bar{E}|+|V|} \ : \ (2), (3) \text{ hold} \right\}$$

$M_+(\text{REP}(G))$

Let us consider inequality (2) for a fixed $v \in V$, and consider a variable x_{ij} for $i \in V$ and $j \in \bar{N}(i) \cup i$

$$x_{ij} \left(\sum_{u \in \bar{N}(v) \cup v} x_{uv} - 1 \right) \qquad \geq 0$$

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$$\left(\sum_{u \in \bar{N}(v) \cup v} x_{ij} x_{uv} - x_{ij} \right) \geq 0$$

$$\sum_{u \in \bar{N}(v) \cup v} x_{ij,uv} - x_{ij} \geq 0$$

$M_+(\text{REP}(G))$

Let us consider inequality (2) for a fixed $v \in V$, and consider a variable x_{ij} for $i \in V$ and $j \in \bar{N}(i) \cup i$

$$(1 - x_{ij}) \left(\sum_{u \in \bar{N}(v) \cup v} x_{uv} \; -1 \right) \qquad \geq 0$$

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$$\left(\sum_{u \in \bar{N}(v) \cup v} x_{uv} - 1 \qquad - \sum_{u \in \bar{N}(v) \cup v} x_{ij} x_{uv} + x_{ij} \right) \geq 0$$

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$$\left(\sum_{u \in \bar{N}(v) \cup v} x_{uv} - 1 - \sum_{u \in \bar{N}(v) \cup v} x_{ij} x_{uv} + x_{ij} \right) \geq 0$$

$$\sum_{u \in \bar{N}(v) \cup v} (x_{uv} - x_{ij,uv}) + x_{ij} \geq 1$$

Repeat this process for all variables and for all constraints in $\mathrm{REP}(G)$

$M_+(\mathrm{REP}(G))$: some remarks

- We assume bound constraints $0 \leq x \leq 1$ are in $Ax \leq b$
- Some constraints generated by $M_+(\cdot)$ may be implied by the PSD condition
- The size of $M_+(\mathrm{REP}(G))$ depends on $|\bar{E}|$, becoming large soon for sparse graphs
- To enhance the practical tractability, we define a relaxation of $M_+(\mathrm{REP}(G))$, denoted by $\hat{M}_+(\mathrm{REP}(G))$, obtained by eliminating some class of inequalities
- Inequalities to be removed selected by preliminary experiments
- Of course we have

$$\hat{M}_+(\mathrm{REP}(G)) \supseteq M_+(\mathrm{REP}(G))$$

$\hat{M}_+(\mathrm{REP}(G))$

$$\min \sum_{u \in V} x_{uu}$$

s.t.

$$\begin{aligned}
&(2.1) \quad && \sum_{u \in \bar{N}(v) \cup v} x_{ij,uv} - x_{ij} \geq 0 && \\
& && && \forall\ v, i \in V,\ j \in \bar{N}(i) \cup i \\
&(2.2) \quad && \sum_{u \in \bar{N}(v) \cup v} (x_{uv} - x_{ij,uv}) + x_{ij} \geq 1 && \\
&(3.1) \quad && x_{ij,uu} - x_{ij,uv} - x_{ij,uw} \geq 0 && \forall\ u, i \in V,\ (v,w) \in G[\bar{N}(u)],\ j \in \bar{N}(i) \cup i \\
&(4) \quad && x_{uv,wj} \geq 0 && \forall\ u, w \in V,\ v \in \bar{N}(u) \cup u,\ j \in \bar{N}(w) \cup w \\
&(5) \quad && x = \mathrm{diag}(X) && \\
&(6) \quad && \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 &&
\end{aligned}$$

Computational Experiments

Comparison among:
$\theta(\bar{G})$, $\hat{M}_+(\mathrm{REP}(G))$ and $\chi^f(G)$.

Solver: `SDPNAL+`[1] (Alternating Direction Method of Multipliers)

Instances:

- DIMACS second implementation challenge's graphs[2]
- Petersen's graph, from the *Kneser graphs* class
- Erdös–Rényi random graphs: $G(n, p)$ is a random graph with n nodes, in which each edge appears with probability p

CPU time limit (solver): 4 hours
Tolerance: 10^{-5}

[1] Yang, Sun, and Toh 2015
[2] David S Johnson and Trick 1996

Computational Experiments: size of $\hat{M}_+(\mathrm{REP}(G))$

Graph	$\|V\|$	$d\%$	n	m
1-FullIns_3	30	23	701	1.166.901
2-Insertions_3	37	11	1.226	2.627.626
3-Insertions_3	56	7	2.917	15.597.685
myciel3	11	36	82	7.534
myciel4	23	28	388	277.867
myciel5	47	22	1.738	9.624.718
petersen	10	33	71	5.671
DSJC125.9	125	90	1.704	15.500.707
r125.1c	125	97	624	423.622
G(40,.5)	40	50	865	3.406.753
G(40,.66)	40	66	603	1.498.981
G(40,.75)	40	75	467	766.105

Computational Experiments

Graph	$\|V\|$	$d\%$	$\theta(\bar{G})$	$\hat{M}_+(\mathrm{REP}(G))$	$\chi^f(G)$	$\chi(G)$
1-FullIns_3	30	23	3.064	†3.294	3.333	4
2-Insertions_3	37	11	2.104	2.434	2.423	4
3-Insertions_3	56	7	2.068	†2.349	2.334	4
myciel3	11	36	2.400	3.036	2.900	4
myciel4	23	28	2.530	3.267	3.245	5
myciel5	47	22	2.639	†3.465	3.553	6
petersen	10	33	2.500	2.720	2.500	3
DSJC125.9	125	90	37.768	†-	42.727	44
r125.1c	125	97	46.000	†-	46.00	46
G(40,.5)	40	50	6.301	†6.277	7.030	8
G(40,.66)	40	66	9.260	†9.126	10.371	11
G(40,.75)	40	75	11.111	11.146	12.030	13

Conclusions & Future works

- We presented a new SDP relaxation for the GCP obtained by the application of $M_+(\cdot)$ operator
- We attempted to handle the size of the SDP by choosing a compact binary formulation of the GCP and then relaxing some class of constraints from $M_+(\mathrm{REP}(G))$
- Even if the state-of-the-art solver for large scale SDPs `SDPNAL+` can handle such formulations, still they are time consuming
- However, computational experiments show that $M_+(\mathrm{REP}(G))$ can yield bounds over $\chi^f(G)$

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- However, computational experiments show that $M_+(\text{REP}(G))$ can yield bounds over $\chi^f(G)$
- The development of dedicated algorithms for solving such SDPs may allow to the resolution of bigger instances, and thus give a clearer picture on their quality

Thanks for your attention!
Questions?

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